Question 1. (12 points) Give a proof of the following statement:

$$(\forall x\, P(x) \to Q(x)) \to ((\exists x\, P(x)) \to \exists y\, Q(y))$$

Number the steps of your proof and indicate what steps are following from what.

Question 2. (12 points) Prove that, for all integers $n \geq 1$, the following equation holds:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Question 3. ($28 = 7 \times 4$ points) Counting problems.

(a) How many ways can a president, vice president, and treasurer be chosen from a group of 11 people?

(b) What is the coefficient of x^3y^5 in $(x+y)^8$?

(c) How many strings of five decimal digits are there that contain exactly three 1s?

(d) How many different functions are there from the set $\{1,2,3,4\}$ to the set $\{1,2,3\}$?

(e) Suppose that every car in a particular parking lot is either a Honda, is red, or is a convertible. How many cars are there in the parking lot if 15 of the cars are Hondas, 20 are red, 11 are convertibles, 5 are red Hondas (2 of which are convertibles), 7 are red convertibles, and 6 are Honda convertibles?

(f) How many non-negative integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 7$ are there?

(g) How many different strings can be made from the letters in ACCESS, using all the letters?

Question 4. (18 = 3 × 6 points) Probabilities.

(a) What is the probability that a five-card poker hand contains a flush, that is, five cards of the same suit? (Express your answer using multiplication, division, and/or combinations; no need to simplify your answer.)

(b) Given a biased coin with $p(\text{heads}) = 2/3$ and $p(\text{tails}) = 1/3$, what is the probability of getting at least one head in four flips?

(c) Given the same coin as in part (b), what is the probability of getting an odd number of heads in four flips?

Question 5. (16 = 4 + 8 + 4 points) Consider the following recurrence relation:

$$\begin{aligned} a_0 &= 0; \\ a_1 &= 1; \\ a_n &= 5a_{n-1} - 6a_{n-2}, \qquad n \geq 2. \end{aligned}$$

(a) What are the values of a_2 and a_3? $a_2 =$ ________ $a_3 =$ ________

(b) Solve this linear recurrence relation to find a general formula for a_n:

(c) Give a big-O solution to the "divide and conquer" recurrence relation $f(n) = 4 \cdot f(n/2) + 2n$.

Question 1. (28 = 2 × 14 points) Proofs: propositional and quantifier.

(a) Give a proof of the following statement:

$$((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r)$$

Number the steps of your proof and indicate what steps are following from what and for what reasons.

(b) Give a proof of the following statement:

$$\exists x\, \forall y\, R(x, y) \to \forall z\, \exists w\, R(w, z)$$

Number the steps of your proof and indicate what steps are following from what and for what reasons.

Question 2. (14 points) Prove by mathematical induction that, for all integers $n \geq 1$, the following equation holds:

$$\sum_{i=1}^{n} (3i - 2) = \frac{n(3n - 1)}{2}$$

Question 3. (24 = 6 × 4 points) Counting problems. Give exact numerical answers.

(a) For a CSci 60 exam, a professor writes 6 multiple choice questions, each with possible answers a, b, or c. If three of the questions have answer a, two of the questions have answer b, and one question has answer c, how many different answer keys are possible, if the questions can be placed in any order?

(b) How many 3-element subsets of a 6-element set are there?

(c) How many ways can 5 cookies be chosen, if there are 3 varieties of cookie (and only the number chosen from each variety matters)?

(d) The total number of students taking at least one of the classes A, B, or C is 175, and each of these three classes has 100 students in it. 60 students are taking A and B at the same time, 50 students are taking A and C at the same time, and 40 students are taking B and C at the same time. How many students are taking all three classes at the same time?

(e) In a standard deck of 52 cards (13 kinds $\times$ 4 suits), how many cards must you deal to be **guaranteed** to have three cards of the same kind?

(f) How many strings of the letters a, b, and c are there that contain 5 letters, exactly 2 of which are as?

Question 4. ($14 = 6 + 4 + 4$ points) Probabilities.

(a) What is the probability that a **seven**-card poker hand contains two three-of-a-kinds? (Express your answer using multiplication, division, and/or combinations; no need to simplify your answer.)

(b) A biased die has $p(1) = p(3) = 1/3$ and $p(2) = p(4) = p(5) = p(6) = 1/12$. What is the probability of rolling an odd number?

(c) If the die from part (b) is rolled 3 times, what is the probability of getting exactly 2 odd numbers?

Question 5. (8 points) Suppose that the sequence a_n satisfies

$$\begin{aligned} a_0 &= 2; \\ a_1 &= 1; \\ a_n &= -10a_{n-1} - 21a_{n-2}, \qquad n \geq 2. \end{aligned}$$

Find a general formula for a_n.